Largest sparse subgraphs of random graphs

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Abstract

For the Erdős-Rényi random graph $G_{n,p}$, we give a precise asymptotic formula for the size $\hat{\alpha}_t(G_{n,p})$ of a largest vertex subset in $G_{n,p}$ that induces a subgraph with average degree at most t, provided that $p = p(n)$ is not too small and $t = t(n)$ is not too large. In the case of fixed t and p, we find that this value is asymptotically almost surely concentrated on at most two explicitly given points. This generalises a result on the independence number of random graphs. For both the upper and lower bounds, we rely on large deviations inequalities for the binomial distribution.

1 Introduction

Given a graph $G = (V, E)$ and a non-negative number t, a vertex subset $S \subseteq V$ is *t-sparse* if the subgraph $G[S]$ induced by S has average degree at most t. The order of a largest such subset is called the *t-sparsity number* of G, denoted $\hat{\alpha}_t(G)$. The t-sparsity number $\hat{\alpha}_t(G)$ is a natural generalisation of the independence number $\alpha(G)$. Recall that an independent set is a vertex subset of G with no edges, i.e. a 0-sparse set; thus the order $\alpha(G)$ of a largest independent set is just $\hat{\alpha}_0(G)$. Note that $\hat{\alpha}_t(G)$ is non-decreasing in terms of t.

We investigate the asymptotic behaviour of $\hat{\alpha}_t(G_{n,p})$, where $G_{n,p}$ is a random graph with vertex set $[n] = \{1, \ldots, n\}$ and each edge is included independently at random with probability p. We focus on fairly dense random graphs: our main result holds when $p = p(n)$ satisfies $p \geq n^{-1/3+\varepsilon}$ for some fixed $\varepsilon > 0$ and p bounded away from 1. We say that a property holds *asymptotically almost surely (a.a.s.)* if it occurs with probability that tends to 1 as $n \to \infty$.

For $t = 0$, that is, the independence number, the asymptotic behaviour in dense random graphs was described forty years ago by Matula [14, 15, 16], Grimmett and McDiarmid [9], and Bollobás and Erdős [4]. For given $0 < p < 1$, define $b = 1/(1-p)$ and

$$\alpha_p(n) = 2\log_b n - 2\log_b \log_b(np) + 2\log_b(e/2) + 1.$$

It was shown that for any $\delta > 0$ a.a.s. $\lfloor \alpha_p(n) - \delta \rfloor \leq \alpha(G_{n,p}) \leq \lfloor \alpha_p(n) + \delta \rfloor$. The main objective of this paper is to provide an analogue of this for $\hat{\alpha}_t(G_{n,p})$.

Some previous estimates on $\hat{\alpha}_t(G_{n,p})$ are implicit in the work of two of the authors. In particular, for fixed p, it was observed using a first moment argument that for any $\varepsilon > 0$, even if t is a growing function of n, as long as $t = o(\ln(np))$, we have $\hat{\alpha}_t(G_{n,p}) \leq (2+\varepsilon)\log_b(np)$ a.a.s., cf. [12, Lemma 2.1]. It follows that $\hat{\alpha}_t(G_{n,p})$ and $\alpha(G_{n,p})$ share the same first-order term growth if $t = o(\ln(np))$. Furthermore, if $t = \omega(\ln(np))$, then $(1-\varepsilon)t/p \leq \hat{\alpha}_t(G_{n,p}) \leq (1+\varepsilon)t/p$, cf. [12, Lemma 2.2]. If $t = \Theta(\ln(np))$, then the growth of the first-order term of $\hat{\alpha}_t(G_{n,p})$ is a multiple of $\log_b(np)$, and large deviation techniques were used to determine the factor (which depends on p and t) [13]. (With the exception of the precise factor at the threshold $t = \Theta(\ln(np))$, these statements have been shown to remain valid for smaller values of p as long as $p \gg 1/n$, cf. [11, Theorem 4.18].)

In this work, we present a sharper description of $\hat{\alpha}_t(G_{n,p})$, using a finer application of the above-mentioned methods. However, we do not concern ourselves with the entire range of choices for the growth of t as a function of n, as above. To get our sharp formula with second- and third-order terms, $p = p(n)$ must not tend to 0 too quickly, and $t = t(n)$ must not grow too quickly. For $0 < p < 1$, define $b = 1/(1-p)$ and

$$\hat{\alpha}_{t,p}(n) = 2\log_b n + (t-2)\log_b\log_b(np) - t\log_b t + t\log_b(2bpe) + 2\log_b(e/2) + 1. \qquad (1)$$

Observe that $\hat{\alpha}_{0,p}(n) = \alpha_p(n)$ (under the convention that $0\ln 0 = 0$) and also $\hat{\alpha}_{t,p}(n) = \alpha_p(n) + t\log_b((2bpe/t)\log_b(np))$. We prove the following.

Theorem 1. *Let $0 < p = p(n) < 1$ be such that p is bounded away from 1 and $p > n^{-1/3+\varepsilon}$, for some positive $\varepsilon < 1/3$. Suppose $t = t(n) \geq 0$ and $\delta = \delta(n) > 0$ satisfy $t = o(\ln n/\ln\ln n)$ and $t^2\ln\ln n/\ln n = o(p\delta)$. Let $\hat{\alpha}_{t,p}(n)$ be as defined in (1). Then $\lfloor\hat{\alpha}_{t,p}(n) - \delta\rfloor \leq \hat{\alpha}_t(G_{n,p}) \leq \lfloor\hat{\alpha}_{t,p}(n) + \delta\rfloor$ a.a.s.*

We see then that $\hat{\alpha}_t(G_{n,p})$ is concentrated around $\hat{\alpha}_{t,p}(n)$ in an interval of width approximately $t^2\ln\ln n/(p\ln n)$. Thus, if $t^2 = o(p\ln n/\ln\ln n)$, then we have *two-point concentration* (or *focusing*), on explicit values.

Let us mention another related generalisation of the independence number. Given a graph $G = (V, E)$ and a non-negative integer t, a vertex subset $S \subseteq V$ is *t-dependent* (or *t-stable*) if the subgraph $G[S]$ induced by S has maximum degree at most t. The order of a largest such subset is called the *t-dependence* (or *t-stability*) number of G, denoted $\alpha_t(G)$. Easily, $\alpha_t(G) \leq \hat{\alpha}_t(G)$. In [8], we considered $\alpha_t(G_{n,p})$, with our attention restricted to fixed p and fixed t, in order to apply analytic techniques to the generating function of degree sequences on k vertices and maximum degree at most t. For $0 < p < 1$, define

$$\alpha_{t,p}(n) = 2\log_b n + (t-2)\log_b\log_b(np) + \log_b(t^t/t!^2) + t\log_b(2bp/e) + 2\log_b(e/2) + 1.$$

We showed in [8] that for any fixed $\delta > 0$, $\lfloor\alpha_{t,p}(n) - \delta\rfloor \leq \alpha_t(G_{n,p}) \leq \lfloor\alpha_{t,p}(n) + \delta\rfloor$ a.a.s. Note that in this setting the difference between the t-sparsity and the t-dependence numbers of $G_{n,p}$ is essentially $\hat{\alpha}_{t,p}(n) - \alpha_{t,p}(n) = 2\log_b(t!e^t/t^t)$. By Stirling's approximation for $t!$ (cf. [3]), we have that $\hat{\alpha}_{t,p}(n) - \alpha_{t,p}(n) \sim \log_b(2\pi t)$ as $t \to \infty$.

We also comment here that, even if t is fixed, the property of t-sparsity is not hereditary, i.e. t-sparsity is not closed under vertex-deletion. Hence the general asymptotic results of Bollobás and Thomason [5] (developed in a long line of research that can be traced back to early results of Alekseev [1], cf. also [2]), for partitions of random graphs according to a fixed hereditary property, are not applicable here. In our previous studies [8, 13], it was useful that t-dependence is hereditary for fixed t. Unfortunately, this is not the case for t-sparsity.

As will become apparent, challenges arise in the second moment computations. We have split this into several parts, according to the degree of overlap between two k-subsets of $[n]$.

Furthermore, in each part we must carefully account for the number of edges which are, say, within one of the k-subsets but not the other, or strictly contained in the overlap, and so on. This careful accounting makes use of large deviations bounds for the binomial distribution.

The term "sparse" may take on a number of different meanings in graph theoretic or algorithmic research. Instead of bounding average degree, one could instead bound for example the degeneracy (i.e. the maximum over all subgraphs of the minimum degree) or the maximum average degree. The counterparts of t-sparsity for these alternative versions of "sparse" are certainly of interest, but we do not pursue them here. We remark only that the counterpart for the former example is bounded below by α_t, while for the latter example it is necessarily bounded between α_t and $\hat{\alpha}_t$.

It is worth noting that the algorithmic complexity of computing the t-sparsity of a graph — for the special cases of t fixed or t parameterised in terms of the order of the target set — was recently studied by Bourgeois et al. [6] and, perhaps unsurprisingly, NP-hardness was shown to hold even in the restricted case of bipartite graphs.

Our paper is organised as follows. In Section 2, we outline the large deviations results that we employ. In Section 3, we perform first moment calculations to obtain Lemma 6; this lemma implies the upper bound in Theorem 1. In Section 4, we give a second moment calculation (Lemma 7) which implies the lower bound in Theorem 1.

2 Large deviations

In this section, we state the large deviations techniques used to precisely describe the average degree of a k-set (a vertex subset of order k) in $G_{n,p}$. For background into large deviations, consult Dembo and Zeitouni [7]; we borrow some notation from this reference. Given $0 < p < 1$, we let $q = 1 - p$ throughout. Also, let

$$\Lambda^*(x) = \begin{cases} x \ln \dfrac{x}{p} + (1-x) \ln \dfrac{1-x}{q} & \text{for } x \in [0,1] \\ \infty & \text{otherwise} \end{cases}$$

(where $\Lambda^*(0) = \ln(1/q)$ and $\Lambda^*(1) = \ln(1/p)$). This is the Fenchel-Legendre transform of the logarithmic moment generating function associated with the Bernoulli distribution with probability p (cf. Exercise 2.2.23(b) of [7]). Some easy calculus verifies that $\Lambda^*(x)$ has a global minimum of 0 at $x = p$, is strictly decreasing on $[0, p)$ and strictly increasing on $(p, 1]$.

In the next lemma — a large deviations result for the binomial distribution — the upper bound follows easily from a strong version of Chernoff's bound, e.g. (2.4) in [10], while the lower bound is implied by a sharp form of Stirling's formula, e.g. (1.4) of [3]: see the appendix of [13] for an explicit proof (when r is integral).

Lemma 2. *There is a constant $\delta > 0$ such that the following holds. Let $0 < p < 1$, let N be a positive integer, and let $X \in \mathrm{Bin}(N, p)$. Then, for each $1 \le r \le N - 1$ such that $r \le Np$,*

$$\delta \cdot \max\left\{r^{-1/2}, (N-r)^{-1/2}\right\} \cdot \exp(-N\Lambda^*(r/N)) \le \mathbb{P}(X \le r) \le \exp(-N\Lambda^*(r/N)).$$

Lemma 2 immediately yields the following estimate on the probability that a given set of size k is t-dependent. For a graph G, we let $\overline{\deg}(G)$ denote the average degree of G.

Lemma 3. *Suppose $0 < p = p(n) < 1$ and suppose that $t = t(n) \ge 1$ and the positive integer $k = k(n)$ satisfy that $t \le p(k-1)$. Then*

(i) $\mathbb{P}(\overline{\deg}(G_{k,p}) \le t) \le \exp\left(-\binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right)\right)$; *and*

(ii) $\mathbb{P}(\overline{\deg}(G_{k,p}) \le t) \ge \exp\left(-\binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right) - \frac{1}{2}\ln k + O(1)\right)$.

For the second moment estimation, we will make use of the following asymptotic calculations, the proofs of which are postponed to the appendix.

Lemma 4. *Suppose $0 < p = p(n) < 1$ and suppose the non-negative number $t = t(n)$ and positive integer $k = k(n)$ satisfy that $t = o(p(k-1))$. For any $\varepsilon = \varepsilon(n)$ with $|\varepsilon| \le 1$,*

$$\Lambda^*\left(\frac{(1+\varepsilon)t}{k-1}\right) = \Lambda^*\left(\frac{t}{k-1}\right) - (1+o(1))\frac{\varepsilon t}{k}\ln\frac{pk}{t}.$$

Lemma 5. *Suppose $0 < p = p(n) < 1$ and that $x = x(n) = o(p)$. Then*

$$\Lambda^*(x) = \ln b\,(1+o(1)).$$

We remark that we will throughout make implicit use of the fact that, for $0 < x < 1$, $-x/(1-x) < \ln(1-x) < -x$.

3 An expectation calculation for the upper bound

In this section, we consider the expected number of t-sparse k-sets. Note that the range of valid values for p in the following lemma is not as restrictive as for Theorem 1, and that the conditions for t and δ are accordingly more general.

Lemma 6. *Let $0 < p = p(n) < 1$ be such that $np \to \infty$ as $n \to \infty$ and p is bounded away from 1. Suppose $t = t(n) \ge 0$ and $\delta = \delta(n) > 0$ satisfy $t = o(\ln(np)/\ln\ln(np))$ and $t^2\log_b \ln(np)/\ln(np) = o(\delta)$. Let $\hat{\alpha}_{t,p}(n)$ be as defined in (1). Let $k^+ = \lceil\hat{\alpha}_{t,p}(n) + \delta\rceil$ and $k^- = \lfloor\hat{\alpha}_{t,p}(n) - \delta\rfloor$ and let $\mathcal{S}_{n,t,k^+}$ and $\mathcal{S}_{n,t,k^-}$ be the collections of t-sparse k^+-sets and k^--sets, respectively. Then*

$$\mathbb{E}(|\mathcal{S}_{n,t,k^-}|) \ge \exp\left((1+o(1))\delta\ln(np)\right) \quad \textit{and}$$
$$\mathbb{E}(|\mathcal{S}_{n,t,k^+}|) \le \exp\left(-(1+o(1))\delta\ln(np)\right).$$

Proof. Note that $\ln b = (1+o(1))p$ if $p \to 0$ as $n \to \infty$. For almost the entire proof, the calculations are carried out in terms of k, instead of k^+ or k^-.

By Lemma 3,

$$\begin{aligned}
\mathbb{E}(|\mathcal{S}_{n,t,k}|) &= \binom{n}{k}\exp\left(-\binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right) + O(\ln k)\right) \\
&= \left(\frac{en}{k}\right)^k \exp\left(-\left(\frac{k-1}{2}\right)\Lambda^*\left(\frac{t}{k-1}\right) + O\left(\frac{\ln k}{k}\right)\right)^k \\
&= \exp\left(1 + \ln n - \ln k - \left(\frac{k-1}{2}\right)\Lambda^*\left(\frac{t}{k-1}\right) + O\left(\frac{\ln k}{k}\right)\right)^k;
\end{aligned}$$

therefore,

$$\frac{2\ln\,\mathbb{E}(|\mathcal{S}_{n,t,k}|)}{k} = 2 + 2\ln n - 2\ln k - (k-1)\Lambda^*\left(\frac{t}{k-1}\right) + O\left(\frac{\ln k}{k}\right). \tag{2}$$

Let us now expand one of the terms in (2) using the formula for Λ^*:

$$\begin{aligned}(k-1)\Lambda^*\left(\frac{t}{k-1}\right) &= t\ln\frac{t}{p(k-1)} + (k-t-1)\ln\left(\left(1-\frac{t}{k-1}\right)\cdot\frac{1}{q}\right)\\ &= t\ln t - t\ln(p(k-1)) + (k-t-1)\ln\left(1-\frac{t}{k-1}\right) + (k-t-1)\ln b.\end{aligned}$$

Since $|t/(k-1)|<1$ for n large enough, we have by Taylor expansion that

$$\begin{aligned}\ln\left(1-\frac{t}{k-1}\right) &= -\frac{t}{k-1} - \frac{t^2}{2(k-1)^2} - \frac{t^3}{3(k-1)^3} - \cdots\,, \text{ and}\\ (k-t-1)\ln\left(1-\frac{t}{k-1}\right) &= -t + \frac{t^2}{2(k-1)} + \frac{t^3}{6(k-1)^2} + \cdots\,,\end{aligned}$$

giving that

$$\begin{aligned}&\frac{2\ln\,\mathbb{E}(|\mathcal{S}_{n,t,k}|)}{k} =\\ &\quad 2 + 2\ln n - 2\ln k - t\ln t + t\ln(p(k-1)) + t - (k-t-1)\ln b + O\left(\frac{t^2+\ln k}{k}\right).\end{aligned} \tag{3}$$

Now, since $t \geq 0$, $np \to \infty$ and $t \leq \ln(np)$ for n large enough, it follows that $k \geq 2\log_b(np) - 2\log_b\ln(np)$ and

$$\begin{aligned}\ln n - \ln k &\leq \ln n - \ln\left(2\log_b(np) - 2\log_b\ln(np)\right)\\ &\leq \ln n - \ln\ln(np) - \ln(2/\ln b) - \ln\left(1-\frac{\ln\ln(np)}{\ln(np)}\right)\\ &\leq \ln n - \ln\ln(np) - \ln(2/\ln b) + O\left(\frac{\ln\ln(np)}{\ln(np)}\right)\end{aligned}$$

for n large enough. Furthermore, for n large enough,

$$\begin{aligned}t\ln(p(k-1)) &\leq t\ln(p(2\log_b(np) + t\log_b\ln(np)))\\ &\leq t\ln\ln(np) + t\ln(2p/\ln b) + t\ln\left(1+\frac{t\ln\ln(np)}{2\ln(np)}\right)\\ &\leq t\ln\ln(np) + t\ln(2p/\ln b) + \frac{t^2\ln\ln(np)}{\ln(np)}.\end{aligned}$$

Similarly, for n large enough,

$$\begin{aligned}\ln n - \ln k &\geq \ln n - \ln\ln(np) - \ln(2/\ln b) + O\left(\frac{t\ln\ln(np)}{\ln(np)}\right) \text{ and}\\ t\ln(p(k-1)) &\geq t\ln\ln(np) + t\ln(2p/\ln b) + O\left(\frac{t\ln\ln(np)}{\ln(np)}\right)\end{aligned}$$

so that

$$\ln n - \ln k = \ln n - \ln\ln(np) - \ln(2/\ln b) + O\left(\frac{t\ln\ln(np)}{\ln(np)}\right) \text{ and} \tag{4}$$

$$t\ln(p(k-1)) = t\ln\ln(np) + t\ln(2p/\ln b) + O\left(\frac{t^2\ln\ln(np)}{\ln(np)}\right). \tag{5}$$

Until here, our calculations did not depend on using k^+ or k^-, but now we have

$$\begin{aligned}&(k^- - t - 1)\ln b \le \\ &2\ln n + (t-2)\ln\ln(np) - (t-2)\ln\ln b - t\ln t + t\ln(2pe) + 2\ln(e/2) \pm \delta\ln b \\ &\le (k^+ - t - 1)\ln b.\end{aligned}$$

Substituting the last inequalities together with (4) and (5) into (3), we obtain, for n large enough,

$$\begin{aligned}\frac{2\ln\,\mathbb{E}(|\mathcal{S}_{n,t,k^-}|)}{k^-} &\ge O\left(\frac{t^2\ln\ln(np)}{\ln(np)}\right) + O\left(\frac{t^2+\ln k}{k}\right) + \delta\ln b = (1+o(1))\delta\ln b \text{ and}\\ \frac{2\ln\,\mathbb{E}(|\mathcal{S}_{n,t,k^+}|)}{k^+} &\le O\left(\frac{t^2\ln\ln(np)}{\ln(np)}\right) + O\left(\frac{t^2+\ln k}{k}\right) - \delta\ln b = -(1+o(1))\delta\ln b,\end{aligned}$$

since $t^2\ln\ln(np)/\ln(np) = o(\delta\ln b)$ and $k \ge \ln(np)$. Now, substituting the expression $(1+o(1))2\log_b(np)$ for k^+ or k^- completes the proof. □

For illustration, let us consider the case of p and t fixed. To satisfy the conditions in the above lemma we need $\delta\ln n/\ln\ln n \to \infty$ as $n \to \infty$. So we may, for instance, set $\delta = (\ln\ln n)^2/\ln n$. We find that the expected number of t-sparse sets of size k^- tends to infinity. The probability that there is a t-sparse set of size at least k^+ is at most $\mathbb{E}(|\mathcal{S}_{n,t,k^+}|) \to 0$ as $n \to \infty$, and so $\hat{\alpha}_t(G_{n,p}) \le \lfloor\hat{\alpha}_{t,p}(n) + \delta\rfloor$ a.a.s.

4 Second moment calculations for the lower bound

Lemma 7. *Let $0 < p = p(n) < 1$ be such that p is bounded away from 1 and $p > n^{-1/3+\varepsilon}$, for some positive $\varepsilon < 1/3$. Suppose $t = t(n) \ge 0$ and $\delta = \delta(n) > 0$ satisfy $t = o(\ln n/\ln\ln n)$ and $t^2\ln\ln n/\ln n = o(p\delta)$. Let $\hat{\alpha}_{t,p}(n)$ be as defined in* (1)*. If $k = k(n) = \lfloor\hat{\alpha}_{t,p}(n) - \delta\rfloor$, then*

$$\mathbb{P}(\hat{\alpha}_t(G_{n,p}) < k) = o(1).$$

Proof. Let $\mathcal{S}_{n,t,k}$ be the collection of t-sparse k-sets in $G_{n,p}$. By Lemma 6,

$$\mathbb{E}(|\mathcal{S}_{n,t,k}|) \ge \exp\left((1+o(1))\delta\ln(np)\right). \tag{6}$$

We use Janson's Inequality (Theorem 2.18(ii) in [10]):

$$\mathbb{P}(\hat{\alpha}_t(G_{n,p}) < k) = \mathbb{P}(|\mathcal{S}_{n,t,k}| = 0) \le \exp\left(-\frac{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)}{\mathbb{E}(|\mathcal{S}_{n,t,k}|) + \Delta}\right), \tag{7}$$

where

$$\Delta = \sum_{A,B\subseteq[n],1<|A\cap B|<k} \mathbb{P}(A, B \in \mathcal{S}_{n,t,k}).$$

We will split this sum into three sums according to the size of $|A \cap B|$ which we denote by ℓ. In particular, let $p(k, \ell)$ be the probability that two k-subsets of $[n]$ that overlap on exactly ℓ vertices are both in $\mathcal{S}_{n,t,k}$. Thus,

$$\Delta = \sum_{\ell=1}^{k-1} \binom{n}{k} \binom{k}{\ell} \binom{n-k}{k-\ell} p(k, \ell).$$

For $\ell \in \{1, \ldots, k-1\}$, let $f(\ell) = \binom{n}{k}\binom{k}{\ell}\binom{n-k}{k-\ell} p(k, \ell)$. We set $\lambda_1 = \varepsilon k/2$ and $\lambda_2 = (1-\varepsilon)k$. (In fact, we shall assume throughout our proof that $\varepsilon < 1/4$; note that this assumption still implies the lemma.) Now we write $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ where the parameters λ_1 and λ_2 determine the ranges of the three sums into which we decompose Δ:

$$\Delta_1 = \sum_{1 \le \ell < \lambda_1} f(\ell), \quad \Delta_2 = \sum_{\lambda_1 \le \ell < \lambda_2} f(\ell), \quad \text{and} \quad \Delta_3 = \sum_{\lambda_2 \le \ell < k} f(\ell).$$

We will show that for $i \in \{1, 2, 3\}$ we have

$$\Delta_i = o\left(\mathbb{E}^2(\mathcal{S}_{n,t,k})\right).$$

So then the result follows from (7).

To bound Δ_i for each $i \in \{1, 2, 3\}$, we consider two arbitrary k-subsets A and B of $[n]$ that overlap on exactly ℓ vertices, i.e. $|A \cap B| = \ell$, and estimate $p(k, \ell)$ by conditioning on the set $E[A \cap B]$ of edges induced by $A \cap B$. In each of the three regimes, we need slightly different techniques to estimate $p(k, \ell)$.

Bounding Δ_1

To bound Δ_1, we write

$$p(k, \ell) = \mathbb{P}(A, B \in \mathcal{S}_{n,t,k}) = \mathbb{P}(A \in \mathcal{S}_{n,t,k} \mid B \in \mathcal{S}_{n,t,k}) \cdot \mathbb{P}(B \in \mathcal{S}_{n,t,k}).$$

The property of having average degree at most t is monotone decreasing, so the conditional probability that $A \in \mathcal{S}_{n,t,k}$ is maximised when $E[A \cap B] = \emptyset$. Thus

$$\begin{aligned}\mathbb{P}(A \in \mathcal{S}_{n,t,k} \mid B \in \mathcal{S}_{n,t,k}) &\le \mathbb{P}(A \in \mathcal{S}_{n,t,k} \mid E[A \cap B] = \emptyset) \\ &\le \frac{\mathbb{P}(A \in \mathcal{S}_{n,t,k})}{\mathbb{P}(E[A \cap B] = \emptyset)} = b^{\binom{\ell}{2}}\, \mathbb{P}(A \in \mathcal{S}_{n,t,k})\end{aligned}$$

implying that $p(k, \ell) \le b^{\binom{\ell}{2}}\, \mathbb{P}^2(A \in \mathcal{S}_{n,t,k})$.

We have though that for n large enough

$$\frac{\binom{k}{\ell}\binom{n-k}{k-\ell}}{\binom{n}{k}} \le 2\frac{\binom{k}{\ell} \cdot n^{k-\ell}/(k-\ell)!}{n^k/k!} = 2\left[\binom{k}{\ell}\right]^2 \frac{\ell!}{n^\ell}.$$

Thus

$$\Delta_1 \le \left(\binom{n}{k}\mathbb{P}(A \in \mathcal{S}_{n,t,k})\right)^2 \left(2\sum_{2 \le \ell < \lambda_1}\left[\binom{k}{\ell}\right]^2 \frac{\ell!}{n^\ell}\, b^{\binom{\ell}{2}}\right).$$

We set

$$s_\ell := \left[\binom{k}{\ell}\right]^2 \frac{\ell!}{n^\ell}\, b^{\binom{\ell}{2}}.$$

Thus we write

$$\Delta_1 \le 2 \cdot \mathbb{E}^2(\mathcal{S}_{n,t,k}) \sum_{2 \le \ell < \lambda_1} s_\ell.$$

We will show that this sum is $o(1)$.

The following claim regards the monotonicity of $\{s_\ell\}$ for ℓ in the range of interest.

Claim 8. *If n is large enough, then for any $2 \le \ell < \lambda_1$ we have $s_{\ell+1}/s_\ell < 1/2$.*

Proof. We have

$$\frac{s_{\ell+1}}{s_\ell} = \frac{(k-\ell)^2}{\ell+1} \frac{b^\ell}{n} \le \frac{k^2}{n} b^{\lambda_1} = O\left(\frac{n^\varepsilon \log^2 n}{np^2}\right),$$

as $b^{\lambda_1} = O(n^\varepsilon)$. But as $p \ge n^{-1/2+\varepsilon}$, we have $np^2 \ge n^{2\varepsilon}$ and, therefore, $s_{\ell+1}/s_\ell < 1/2$, for large enough n. $\square$

Thus the sum $\sum_{\ell<\lambda_1} s_\ell$ is essentially determined by its first term s_2:

$$\sum_{\ell<\lambda_1} s_\ell \le 2s_2.$$

But we have

$$s_2 = O\left(\frac{k^4}{n^2}\right) = O\left(\frac{\log^4 n}{n^2p^4}\right) = O\left(n^2 \frac{\log^4 n}{(np)^4}\right) = o(1),$$

if $p \ge n^{-1/2+\varepsilon}$.

Bounding Δ_2

The bound on $\Delta_2 = \sum_{\lambda_1 \le \ell < \lambda_2} f(\ell)$ involves a more thorough consideration of the number of edges in the overlap between the sets A and B.

Let us fix some integer ℓ such that $\lambda_1 \le \ell < \lambda_2$. We will show that $f(\ell)/\mathbb{E}^2(|\mathcal{S}_{n,t,k}|) = o(1/k)$. With A, B being two sets of vertices, each having size k, that overlap on ℓ vertices, we have

$$\frac{f(\ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} = \frac{\binom{n-k}{k-\ell}\binom{k}{\ell}}{\binom{n}{k}} \frac{\mathbb{P}(A, B \in \mathcal{S}_{n,t,k})}{\mathbb{P}^2(A \in \mathcal{S}_{n,t,k})}. \tag{8}$$

The first ratio on the right-hand side can be bounded for n sufficiently large as follows:

$$\frac{\binom{k}{\ell}\binom{n-k}{k-\ell}}{\binom{n}{k}} \le 2^{k+1} \frac{n^{k-\ell}/(k-\ell)!}{n^k/k!} \le 2^{k+1}\binom{k}{\ell}\frac{\ell!}{n^\ell} \le 2^{2k+1}\left(\frac{k}{n}\right)^\ell. \tag{9}$$

We now give estimates on $\mathbb{P}(A, B \in \mathcal{S}_{n,t,k})$ as well as on $\mathbb{P}(A \in \mathcal{S}_{n,t,k})$. For each set A of vertices, let $E[A]$ denote the set of edges with both their endvertices in A, and let

$e(A) = |E[A]|$. Also, let $e'(A,B) = e(A) - e(A \cap B)$, the number of edges in $E[A] \setminus E[A \cap B]$. Setting $I = A \cap B$, we have

$$\mathbb{P}(A, B \in \mathcal{S}_{n,t,k}) \leq \mathbb{P}(e(I) \leq kt/2) \cdot \mathbb{P}^2(e'(A,B) \leq kt/2).$$

We will bound the two probabilities on the right-hand side of the above inequality using Lemma 2. As $e(I) \in \mathrm{Bin}\left(\binom{\ell}{2}, p\right)$ and $e'(A,B) \in \mathrm{Bin}\left(\binom{k}{2} - \binom{\ell}{2}, p\right)$, with $x_I = kt/(\ell(\ell-1))$ and $x_{A,B} = kt/(k(k-1) - \ell(\ell-1))$ we have

$$\begin{aligned}\mathbb{P}(e(I) \leq kt/2) &= \exp\left(-\binom{\ell}{2}\Lambda^*(x_I) + O(\ln k)\right)\\ \mathbb{P}(e'(A,B) \leq kt/2) &= \exp\left(-\left(\binom{k}{2} - \binom{\ell}{2}\right)\Lambda^*(x_{A,B}) + O(\ln k)\right).\end{aligned}$$

Now, note that both x_I and $x_{A,B}$ are $o(p)$. This holds since $x_I, x_{A,B} = O(t/k)$ and $k = \Theta(\ln n/p)$ and $t = o(\ln n/\ln\ln n)$. But now we can apply Lemma 5 to obtain

$$\begin{aligned}&\mathbb{P}(e(I) \leq kt/2) \cdot \mathbb{P}^2(e'(A,B) \leq kt/2)\\ &= \exp\left(-\binom{\ell}{2}\ln b(1+o(1)) - 2\left(\binom{k}{2} - \binom{\ell}{2}\right)\ln b(1+o(1)) + O(\ln k)\right)\\ &= \exp\left(\binom{\ell}{2}\ln b - 2\binom{k}{2}\ln b + o(k^2p)\right).\end{aligned} \tag{10}$$

Similarly,

$$\mathbb{P}(A \in \mathcal{S}_{n,t,k}) = \exp\left(-\binom{k}{2}\ln b + o(k^2p)\right). \tag{11}$$

Hence the estimates in (10) and (11) yield

$$\frac{\mathbb{P}(A, B \in \mathcal{S}_{n,t,k})}{\mathbb{P}^2(A \in \mathcal{S}_{n,t,k})} = \exp\left(\binom{\ell}{2}\ln b + o(k^2p)\right).$$

Now, combining the above together with (9) and the right-hand side of (8), we obtain

$$\begin{aligned}\frac{f(\ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} &= \exp\left(-\ell\ln n + \ell\ln k + \binom{\ell}{2}\ln b + o(k^2p)\right)\\ &= \exp\left(-\ell\left(\ln n - \ln k - \frac{\ell\ln b}{2} + o(kp)\right)\right).\end{aligned} \tag{12}$$

We will show that $\ln n - \ln k - \ell\ln b/2 \to \infty$ as $n \to \infty$, for any $\lambda_1 \leq \ell < \lambda_2$. Recall that $k = (2+o(1))\log_b(np)$. Thus $\ln n - \ln k = \ln(n\ln b/(2\ln(np))) + o(1) \geq \ln(np) + O(\ln\ln n)$ as $\ln b \geq p$. Also, as $\ell < (1-\varepsilon)k$, we have $\ell\ln b/2 < (1-\varepsilon)\ln(np)(1+o(1))$. Therefore

$$\ln n - \ln k - \frac{\ell\ln b}{2} > \varepsilon\ln(np) + o(\ln n).$$

These two bounds substituted into (12) now imply that

$$\frac{f(\ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} = \exp\left(-\Omega(\ell\ln n)\right), \tag{13}$$

uniformly for all $\lambda_1 \leq \ell < \lambda_2$. But since $\ell \geq \varepsilon k/2$, this bound is $o(1/k)$ and therefore $\Delta_2 = o\left(\mathbb{E}^2(\mathcal{S}_{n,t,k})\right)$.

Bounding Δ_3

Next, to bound Δ_3, the aim here is also to show that for $\ell \geq \lambda_2$ we have

$$\frac{f(\ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} = o\left(\frac{1}{k}\right).$$

This is the portion of Δ that is the most difficult to control. It is also the regime in which the condition $p \geq n^{-1/3+\varepsilon}$ is required. (We only required the weaker condition $p \geq n^{-1/2+\varepsilon}$ to bound Δ_1 and Δ_2.) In this regime, we need to separately treat two sub-regimes which are divided according to the edge count in the overlap.

Let us consider an arbitrary $\ell \geq \lambda_2$ and write

$$p(k,\ell) = \sum_{m=0}^{\lfloor tk/2 \rfloor} p(k,\ell,m)$$

where $p(k,\ell,m) = \mathbb{P}(A, B \in \mathcal{S}_{n,t,k} \wedge e(A \cap B) = m)$. (Note that $m \leq tk/2$ or trivially both $A, B \notin \mathcal{S}_{n,t,k}$.) We split this summation in two:

$$p(k,\ell) = \sum_{m=0}^{\mu} p(k,\ell,m) + \sum_{m=\mu+1}^{\lfloor tk/2 \rfloor} p(k,\ell,m) =: p_1(k,\ell) + p_2(k,\ell), \tag{14}$$

with $\mu = \max\{0, \lfloor tk/2 - (k-\ell)(k+\ell-1)\psi p/2 \rfloor\}$, where ψ is the unique $0 < \psi = \psi(n) < 1$ such that $\Lambda^*(\psi p) = (1-\xi)\ln b$, for some fixed $0 < \xi < 1$ yet to be specified.

That ψ exists is guaranteed by the fact that Λ^* is strictly decreasing on $[0,p)$, $\Lambda^*(0) = \ln b$ and $\Lambda^*(p) = 0$. We now show that ψ is bounded away from 0. We have that ψ satisfies

$$\begin{aligned}
\xi = 1 - \frac{\Lambda^*(\psi p)}{\ln b} &= 1 - \left(\frac{\psi p}{\ln b}\ln\psi + \frac{1-\psi p}{\ln b}\ln\frac{1-\psi p}{q}\right) \\
&= \psi p + \frac{\psi p}{\ln b}\ln\frac{1}{\psi} - \frac{(1-\psi p)\ln(1-\psi p)}{\ln b} \\
&\leq \psi p + \frac{\psi p}{\ln b}\ln\frac{1}{\psi} + \frac{\psi p}{\ln b}
\end{aligned}$$

since $(1-x)\ln(1-x) \geq -x$ for $0 < x < 1$. Thus, using also $p \leq \ln b$,

$$\xi \leq \psi p + \psi \ln\frac{1}{\psi} + \psi \leq \psi(2 + \ln\psi)$$

But $x(2 + \ln x) \to 0$ as $x \searrow 0$. Hence there exists $\delta = \delta(\xi) > 0$ such that $\psi \geq \delta$ uniformly over p.

Let us give a bound on $p_1(k,\ell)$. We may assume that $(k-\ell)(k+\ell-1)\psi p \leq tk$, or else the sum is empty. It will suffice to consider $E[A \cap B]$ alone. Observe that $e(A \cap B)$ is binomially distributed with parameters $\binom{\ell}{2}$ and p. But $\ell \geq \lambda_2 = \Omega(\ln n/p)$, and so since $t = o(\ln n)$ it follows that $\mu \leq tk/2 = o\left(p\binom{\ell}{2}\right)$. Thus, by Lemma 2,

$$\begin{aligned}
p_1(k,\ell) \leq \mathbb{P}(e(A \cap B) \leq \mu) &\leq \exp\left(-\binom{\ell}{2}\Lambda^*\left(\mu \middle/ \binom{\ell}{2}\right)\right) \\
&= \exp\left(-\binom{\ell}{2}\Lambda^*\left(\frac{tk}{\ell(\ell-1)} - \frac{(k-\ell)(k+\ell-1)\psi p}{\ell(\ell-1)}\right)\right).
\end{aligned}$$

By Lemma 4, since $tk = o(p\ell(\ell-1))$ and $0 \le (k-\ell)(k+\ell-1)\psi p \le tk$,

$$\begin{aligned} p_1(k,\ell) &\le \exp\left(-\binom{\ell}{2}\left(\Lambda^*\left(\frac{tk}{\ell(\ell-1)}\right) + (1+o(1))\frac{(k-\ell)(k+\ell-1)\psi p}{\ell(\ell-1)}\ln\frac{p\ell(\ell-1)}{tk}\right)\right) \\ &= \exp\left(-\binom{\ell}{2}\Lambda^*\left(\frac{tk}{\ell(\ell-1)}\right) - (1+o(1))\left(\frac{(k-\ell)(k+\ell-1)}{2}\psi p\ln\frac{p\ell(\ell-1)}{tk}\right)\right). \end{aligned} \tag{15}$$

To estimate $p_2(k,\ell)$, we need a finer argument in which we also consider the sets $E[A]$ and $E[B]$ of edges induced by A and B, respectively. In particular, let X_1 and X_2 denote $e'(A,B)$ (recall that this is $e(A) - e(A \cap B)$) and $e'(B,A)$, respectively. Note that X_1 and X_2 are binomially distributed with parameters $\ell(k-\ell) + \binom{k-\ell}{2} = (k-\ell)(k+\ell-1)/2$ and p. Furthermore, X_1 and X_2 and $e(A\cap B)$ are independent. Therefore,

$$p_2(k,\ell) \le \mathbb{P}(e(A\cap B) \le tk/2)\cdot \mathbb{P}^2(X_1 \le tk/2 - \mu - 1).$$

By Lemma 2, (since $tk/2 = o\left(p\binom{\ell}{2}\right)$,)

$$\mathbb{P}(e(A\cap B) \le tk/2) \le \exp\left(-\binom{\ell}{2}\Lambda^*\left(\frac{tk}{\ell(\ell-1)}\right)\right)$$

and (as $0 < \psi < 1$)

$$\begin{aligned} \mathbb{P}(X_1 \le tk/2 - \mu - 1) &\le \mathbb{P}\left(X_1 \le \frac{(k-\ell)(k+\ell-1)}{2}\psi p\right) \\ &\le \exp\left(-\frac{(k-\ell)(k+\ell-1)}{2}\Lambda^*(\psi p)\right) \\ &= \exp\left(-\frac{(k-\ell)(k+\ell-1)}{2}(1-\xi)\ln b\right). \end{aligned}$$

We conclude that

$$p_2(k,\ell) \le \exp\left(-\binom{\ell}{2}\Lambda^*\left(\frac{tk}{\ell(\ell-1)}\right) - \frac{(k-\ell)(k+\ell-1)}{2}(2-2\xi)\ln b\right). \tag{16}$$

Comparing with (15), since $tk = o(p\ell(\ell-1))$ and $\psi = \Theta(1)$, we notice that $p_1(k,\ell)$ is asymptotically smaller than the above upper bound on $p_2(k,\ell)$.

Now, from $\ell \ge \lambda_2$ it follows that

$$\ell(\ell-1) \ge (k-\ell)(k+\ell-1).$$

Indeed, $(k-\ell)(k+\ell-1) \le k^2 - \ell^2 \le k^2 - (1-\varepsilon)^2k^2 \le 2\varepsilon k^2$ and also for n sufficiently large $\ell(\ell-1) \ge (1-\varepsilon)^2k^2 \ge (1-2\varepsilon)k^2$. As $\varepsilon < 1/4$, the above inequality holds.

Thus, since $t = o(p(k-1))$, we obtain using Lemma 4 that

$$\begin{aligned} \binom{\ell}{2}\Lambda^*\left(\frac{tk}{\ell(\ell-1)}\right) &= \binom{\ell}{2}\Lambda^*\left(\left(1+\frac{(k-\ell)(k+\ell-1)}{\ell(\ell-1)}\right)\frac{t}{k-1}\right) \\ &= \binom{\ell}{2}\Lambda^*\left(\frac{t}{k-1}\right) - (1+o(1))\binom{\ell}{2}\frac{(k-\ell)(k+\ell-1)t}{\ell(\ell-1)k}\ln\frac{pk}{t} \\ &= \binom{\ell}{2}\Lambda^*\left(\frac{t}{k-1}\right) - (1+o(1))\frac{(k-\ell)(k+\ell-1)}{2}p\frac{\ln(pk/t)}{pk/t} \\ &= \binom{\ell}{2}\Lambda^*\left(\frac{t}{k-1}\right) - o\left(\frac{(k-\ell)(k+\ell-1)}{2}\ln b\right). \end{aligned} \tag{17}$$

Furthermore, since Λ^* is strictly decreasing on $[0, p)$ and $\Lambda^*(0) = \ln b$,

$$\begin{aligned}
\binom{\ell}{2}\Lambda^*\left(\frac{t}{k-1}\right) &= \left(\binom{k}{2} - \frac{(k-\ell)(k+\ell-1)}{2}\right)\Lambda^*\left(\frac{t}{k-1}\right) \\
&= \binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right) - \frac{(k-\ell)(k+\ell-1)}{2}\Lambda^*\left(\frac{t}{k-1}\right) \\
&\geq \binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right) - \frac{(k-\ell)(k+\ell-1)}{2}\ln b. \qquad (18)
\end{aligned}$$

Combining (16)–(18), we conclude that

$$p_2(k,\ell) \leq \exp\left(-\binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right) - (1+o(1))\frac{(k-\ell)(k+\ell-1)}{2}(1-2\xi)\ln b\right). \qquad (19)$$

As remarked earlier, $p_1(k,\ell)$ is asymptotically smaller than the upper bound for $p_2(k,\ell)$. Hence it suffices to show that

$$\frac{\binom{n}{k}\binom{k}{\ell}\binom{n-k}{k-\ell}\, p_2(k,\ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} = o\left(\frac{1}{k}\right).$$

Recall that with A being a set of vertices of size k we have

$$\mathbb{E}(|\mathcal{S}_{n,t,k}|) = \binom{n}{k}\mathbb{P}(A \in \mathcal{S}_{n,t,k}) = \binom{n}{k}\ \exp\left(-\binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right) + O(\ln k)\right),$$

where the last equality follows from Lemma 3. Thus, using (19), we have

$$\begin{aligned}
&\frac{\binom{n}{k}\binom{k}{\ell}\binom{n-k}{k-\ell}\, p_2(k,\ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} = \\
&\frac{\binom{k}{\ell}\binom{n-k}{k-\ell}}{\mathbb{E}(|\mathcal{S}_{n,t,k}|)}\ \exp\left(-(1+o(1))\frac{(k-\ell)(k+\ell-1)}{2}(1-2\xi)\ln b + O(\ln k)\right).
\end{aligned} \qquad (20)$$

Now

$$\binom{k}{\ell}\binom{n-k}{k-\ell} \leq (kn)^{k-\ell}.$$

Thus using the lower bound on $\mathbb{E}(|\mathcal{S}_{n,t,k}|)$ given in (6) we obtain

$$\begin{aligned}
&\ln\frac{\binom{n}{k}\binom{k}{\ell}\binom{n-k}{k-\ell}\, p_2(k,\ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} = \\
&(k-\ell)\ln(kn) - (1+o(1))\delta\ln(np) - (1+o(1))\frac{(k-\ell)(k+\ell)}{2}(1-2\xi)\ln b + O(\ln k).
\end{aligned} \qquad (21)$$

Now, we have for n sufficiently large

$$\begin{aligned}
(k-\ell)\ln(kn) &- (1+o(1))\frac{(k-\ell)(k+\ell)}{2}(1-2\xi)\ln b \\
&\leq (k-\ell)\left(\ln(nk) - \frac{(2-\varepsilon)k}{2}(1-2\xi)\ln b\right) \\
&\leq (k-\ell)(\ln(nk) - (2-2\varepsilon)\ln(np)),
\end{aligned}$$

where in the last inequality we used a choice of ξ small enough as well as the fact that $k \ln b = (1 + o(1)) 2 \ln(np)$. But also $k \leq (1 + o(1)) 2 \ln(np)/p$, as $\ln b \geq p$. This implies that $\ln k \leq \ln \ln(np) - \ln p + O(1)$. Hence, for n sufficiently large,

$$
\begin{aligned}
(k - \ell) \ln(kn) - (1 + o(1)) & \frac{(k - \ell)(k + \ell)}{2} (1 - 2\xi) \ln b \\
& \leq (k - \ell)(-\ln n + \ln \ln(np) + O(1) - 3 \ln p + 2\varepsilon \ln(np)) \\
& \leq (k - \ell)(-\ln n - 3 \ln n^{-1/3+\varepsilon} + 3\varepsilon \ln(np)) \leq 0,
\end{aligned}
$$

where we used the condition $p \geq n^{-1/3+\varepsilon}$ in the second last inequality. Substituting this into (21), we obtain

$$
\frac{\binom{n}{k} \binom{k}{\ell} \binom{n-k}{k-\ell} p_2(k, \ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} \leq \exp\left(-(1 + o(1))\delta \ln(np) + O(\ln k)\right).
$$

But $\ln \ln n / \ln n = o(p\delta)$ and therefore $\ln \ln n/p = o(\delta \ln(np))$. On the other hand, $\ln k = O\left(\ln(\ln n/p)\right)$, which implies that $\ln k = o(\delta \ln(np))$. Therefore

$$
\frac{\binom{n}{k} \binom{k}{\ell} \binom{n-k}{k-\ell} p_2(k, \ell)}{\mathbb{E}^2(|\mathcal{S}_{n,t,k}|)} = o\left(\frac{1}{k}\right),
$$

as required. □

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A Appendix

Proof of Lemma 4. We split the proof into two cases. First, if $\varepsilon = -1$, then

$$
\begin{aligned}
&\Lambda^*\left(\frac{t}{k-1}\right) - \Lambda^*\left(\frac{(1+\varepsilon)t}{k-1}\right) = \Lambda^*\left(\frac{t}{k-1}\right) - \Lambda^*(0)\\
&= \frac{t}{k-1}\ln\frac{t}{p(k-1)} + \left(1-\frac{t}{k-1}\right)\ln\frac{1-\frac{t}{k-1}}{q} - \ln\frac{1}{q}\\
&= \frac{t}{k-1}\ln\frac{t}{p(k-1)} + \left(1-\frac{t}{k-1}\right)\ln\left(1-\frac{t}{k-1}\right) - \frac{t}{k-1}\ln\frac{1}{q}\\
&= \frac{t}{k-1}\ln\frac{qt}{p(k-1)} - \frac{t}{k-1} + O\left(\frac{t^2}{k^2}\right) = -(1+o(1))\frac{t}{k}\ln\frac{pk}{t}\\
&= (1+o(1))\frac{\varepsilon t}{k}\ln\frac{pk}{t}
\end{aligned}
$$

(where we used $t = o(k)$ and the Taylor expansion of $(1 - t/(k-1))\ln(1 - t/(k-1))$). Otherwise, $-1 < \varepsilon \le 1$ and

$$\begin{aligned}
\Lambda^*\left(\frac{(1+\varepsilon)t}{k-1}\right) &= \left(\frac{(1+\varepsilon)t}{k-1}\right)\ln\frac{(1+\varepsilon)t}{p(k-1)} + \left(1 - \frac{(1+\varepsilon)t}{k-1}\right)\ln\frac{k-1-(1+\varepsilon)t}{q(k-1)} \\
&= \frac{t}{k-1}\ln\frac{(1+\varepsilon)t}{p(k-1)} + \frac{\varepsilon t}{k-1}\ln\frac{(1+\varepsilon)t}{p(k-1)} \\
&\quad + \left(1 - \frac{t}{k-1}\right)\ln\frac{k-1-(1+\varepsilon)t}{q(k-1)} - \frac{\varepsilon t}{k-1}\ln\frac{k-1-(1+\varepsilon)t}{q(k-1)} \\
&= \Lambda^*\left(\frac{t}{k-1}\right) + \frac{t}{k-1}\ln(1+\varepsilon) + \left(1 - \frac{t}{k-1}\right)\ln\left(1 - \frac{\varepsilon t}{k-1-t}\right) \\
&\quad + \frac{\varepsilon t}{k-1}\ln\frac{q(1+\varepsilon)t}{p(k-1-(1+\varepsilon)t)}
\end{aligned}$$

and the lemma follows by observing that, by Taylor expansion,

$$\begin{aligned}
\frac{t}{k-1}\ln(1+\varepsilon) + \left(1 - \frac{t}{k-1}\right)\ln\left(1 - \frac{\varepsilon t}{k-1-t}\right) &= O\left(\frac{\varepsilon^2 t}{k}\right) \text{ and} \\
\frac{\varepsilon t}{k-1}\ln\frac{q(1+\varepsilon)t}{p(k-1-(1+\varepsilon)t)} &= -(1+o(1))\frac{\varepsilon t}{k}\ln\frac{pk}{t}.
\end{aligned}$$

□

Proof of Lemma 5. Since $(1-x)\ln(1-x) = O(x)$ as $x \to 0$,

$$\begin{aligned}
\Lambda^*(x) &= x\ln\left(\frac{x}{p}\right) + (1-x)\ln b + (1-x)\ln(1-x) \\
&= \ln b\left(1 + \frac{x}{\ln b}\ln\left(\frac{x}{p}\right) + O\left(\frac{x}{\ln b}\right)\right).
\end{aligned}$$

But $p = \Theta(\ln b)$ and $x = o(p)$, and the lemma follows. □